Optimal Fusion of a Given Quaternion with Vector Measurements

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Introduction

SEVERAL satellites that use devices called autonomous star trackers (ASTs) are presently operating. These devices put out the satellites' attitude in the form of a quaternion. Usually the satellites also carry other attitude measuring devices, such as sun sensors and magnetometers that measure vectors in body coordinates. Although the accuracy of the AST surpasses that of the other sensors, due to the synergistic effect of sensor fusion, it is still desirable to incorporate the measurements of the less accurate sensors in the attitude determination process. The question is, then, how to blend optimally the AST-generated quaternion with vector measurements. This problem rose, for example, in the design of the attitude determination algorithm of the Microwave Anisotropy Probe (MAP) satellite, which was launched on 30 June 2001. MAP has two ASTs and two sun sensors, one of which is more accurate than the other. Although we have two quaternion-generating devices and two vector-measuring sensors, we consider here only one of each. The extension of the solution, proposed here, to multiple devices and multiple sensors, is immediate.

We note that the quaternion is a four-element vector that yields the whole attitude, whereas vector-measuring sensors yield three-dimensional vectors each containing only partial information on the attitude. Therefore, we cannot cast the problem of optimal attitude determination in the form of Wahba's problem.[1] That is, we cannot blend quaternions with vector measurements using known algorithms.

If we have more than one simultaneous vector measurement, we can use the vector measurements to, first, find attitude expressed in quaternion form and, then, blend this quaternion with the given one. However, when we have only one vector measurement, this is not possible. Therefore, we need an algorithm that can blend the given quaternion even with one vector measurement.

The algorithm presented here consists of two steps. In the first step, the quaternion is converted into a pair of pseudovector measurements that express the attitude, and then, in the second step, these pseudovector measurements, together with the given vector measurement (or measurements), are used as inputs to the q-method algorithm,[2] which generates the optimal quaternion. The resultant quaternion is optimal in the sense that it is the best fit, in the least-squares sense, to all of the vectors.

Algorithm

As mentioned, in the first step the given quaternion is converted into two vectors that represent an equivalent attitude. We do not want the two vectors to be collinear, and because the farthest away from this situation is the case where the two are perpendicular to one another, we pose it as a requirement. Because we still have more freedom in determining these vectors, we require that none of them be collinear with the new vector measurement, even though this constraint is not necessary. To meet these conditions, we choose two vectors that are perpendicular to one another and to the vector measurement.

The orthogonality of the three vectors can be achieved in the following way. Although we can choose any vector to start the process to be described, for simplicity we choose a unit vector along one of the reference axes, that is, we choose either $\boldsymbol{e}_1^T = [1 \;\; 0 \;\; 0]$, $\boldsymbol{e}_2^T = [0 \;\; 1 \;\; 0]$, or $\boldsymbol{e}_3^T = [0 \;\; 0 \;\; 1]$. To decide which of these vectors to select as a starting vector, we compute the norm of the cross product of $\boldsymbol{r}_3$, a unit vector in the direction of the measured vector expressed in the reference coordinates, and each one of the three, and select that vector that produces the largest norm. In other words, we compute t_i for $i = 1, 2, 3$ where

$$t_i = \max_i\{|\boldsymbol{e}_i \times \boldsymbol{r}_3|\} \tag{1}$$

and then we consider the largest t_i. We denote by $i_{\max}$ the i that corresponds to this t_i and by $\boldsymbol{e}_{i_{\max}}$ the unit vector that corresponds to this t_i. We then choose $\boldsymbol{r}_1$ as follows:

$$\boldsymbol{r}_1 = \frac{\boldsymbol{e}_{i_{\max}} \times \boldsymbol{r}_3}{\left|\boldsymbol{e}_{i_{\max}} \times \boldsymbol{r}_3\right|} \tag{2}$$

This choice assures that the normalization carried out in the last equation implies a division by the largest possible number, which, of course, is the most numerically advantageous case. It is obvious that this $\boldsymbol{r}_1$ is perpendicular to $\boldsymbol{r}_3$. We then generate the second necessary vector $\boldsymbol{r}_2$ through the following cross-product operation:

$$\boldsymbol{r}_2 = \boldsymbol{r}_3 \times \boldsymbol{r}_1 \tag{3}$$

Clearly, the triad $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, and $\boldsymbol{r}_3$ is an orthogonal one. This meets the two aforementioned requirements.

Next we need to calculate the two reference vectors, $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, that correspond to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. (Note that $\boldsymbol{b}_3$ is our new measurement.) These two vectors are determined by the attitude matrix, which corresponds to the given quaternion. The corresponding attitude matrix that transforms from the reference to the body coordinates is as follows[3]:

$$D_b^r = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_2 + q_3q_4) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_1q_4) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix} \tag{4}$$

then

$$\boldsymbol{b}_1 = D_b^r \boldsymbol{r}_1 \tag{5a}$$

$$\boldsymbol{b}_2 = D_b^r \boldsymbol{r}_2 \tag{5b}$$

Next we assign a normalized weight to each of the two pairs, $\boldsymbol{r}_1\,\boldsymbol{b}_1$ and $\boldsymbol{r}_2\,\boldsymbol{b}_2$, as well as to the pair $\boldsymbol{r}_3\,\boldsymbol{b}_3$. Because the quaternion was obtained using two star tracker measurements, we assign to each one of the pairs $\boldsymbol{r}_1\,\boldsymbol{b}_1$ and $\boldsymbol{r}_2\,\boldsymbol{b}_2$ the same sigma value σ_{ST} that corresponds to a star tracker measurement error. Similarly, we assign a sigma value σ_3 to the third pair, which corresponds to the measurement error of the sensor that measures $\boldsymbol{b}_3$. The nonnormalized weights of the three vectors are then computed as

$$\alpha_1 = \alpha_2 = 1\big/\sigma_{\mathrm{ST}}^2 \tag{6a}$$

$$\alpha_3 = 1\big/\sigma_3^2 \tag{6b}$$

and the normalized weight, which will be used in the q-method, is computed as

$$a_i = \alpha_i \bigg/ \sum_{i=1}^{3} \alpha_i \tag{7}$$

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The extension of this case to the case where the AST star trackers differ in their accuracy from one another is rather trivial.

We are now ready to perform the second step of the algorithm, namely, to use the q-method to compute the optimal quaternion that takes in account both the AST-generated quaternion and the new vector measurement. We first compute

$$B = \sum_{i=1}^{3} a_i \boldsymbol{b}_i \boldsymbol{r}_i^T \tag{8a}$$

$$S = B + B^T \tag{8b}$$

$$\boldsymbol{z} = \sum_{i=1}^{n} a_i (\boldsymbol{b}_i \times \boldsymbol{r}_i) \tag{8c}$$

$$s = \sum_{i=1}^{3} a_i \boldsymbol{b}_i^T \boldsymbol{r}_i \tag{8d}$$

and then use these values to compute the K matrix of the the q-method as follows:

$$K = \left[\begin{array}{c|c} S - \sigma I_3 & \boldsymbol{z} \\ \hline \boldsymbol{z}^T & s \end{array} \right] \tag{9}$$

Then the sought quaternion $\boldsymbol{q}^*$ is the eigenvector that corresponds to the largest eigenvalue of K.

When the logic of the algorithm presented here is used, it is easy to extend it to a case where there is more than one AST and more than one additional measurement. The extension is obvious and needs not be repeated here.

Example

In this example we generated the following correct inertial to body attitude matrix

$$D_0 = \begin{bmatrix} 0.385 & 0.352 & 0.853 \\ -0.616 & 0.786 & -0.046 \\ -0.687 & -0.508 & 0.519 \end{bmatrix} \tag{10}$$

In order to simulate the quaternion which was determined by the AST for this attitude, we arbitrarily assumed that this attitude was determined by the measurement of two unit vectors whose expression in inertial coordinates was

$$\boldsymbol{u}_1 = \begin{bmatrix} 0.267 \\ 0.535 \\ 0.802 \end{bmatrix} \tag{11a}$$

$$\boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 0.707 \\ -0.707 \end{bmatrix} \tag{11b}$$

To make sure that these vectors were not nearly collinear, we checked and found that the angle between them was 79.1 deg. Using D_0, we obtained these vectors in body coordinates as

$$\boldsymbol{v}_1 = \begin{bmatrix} 0.975 \\ 0.219 \\ -0.039 \end{bmatrix} \tag{11c}$$

$$\boldsymbol{v}_2 = \begin{bmatrix} -0.355 \\ 0.588 \\ -0.727 \end{bmatrix} \tag{11d}$$

We then assumed that the vectors were measured by the star trackers of the AST, which introduced measurement errors. The measured vectors were computed as follows. First an error of $\pm\varepsilon$ was added to each component such that

$$\boldsymbol{v}_1' = \boldsymbol{v}_1 + \begin{bmatrix} \pm\varepsilon \\ \pm\varepsilon \\ \pm\varepsilon \end{bmatrix} \tag{11e}$$

$$\boldsymbol{v}_2' = \boldsymbol{v}_2 + \begin{bmatrix} \pm\varepsilon \\ \pm\varepsilon \\ \pm\varepsilon \end{bmatrix} \tag{11f}$$

where $\varepsilon = 25$ arc-s. Then $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2'$ were normalized, which resulted in the vector measurements $\boldsymbol{v}_{1,\text{meas}}$ and $\boldsymbol{v}_{2,\text{meas}}$ respectively. Because we assumed that both vectors were measured with the same accuracy, the weights a_1 and a_2 were chosen to be one-half each. These weights were assigned to the pairs $\boldsymbol{v}_{1,\text{meas}}$ and $\boldsymbol{u}_1$ and $\boldsymbol{v}_{2,\text{meas}}$ and $\boldsymbol{u}_2$, which where then processed by the q-method of Eqs. (8) and (9) to yield the simulated AST output quaternion $\boldsymbol{q}_{\text{AST}}$. A figure of merit that represented the accuracy of the AST-determined direction cosine matrix was then computed as follows. The quaternion $\boldsymbol{q}_{\text{AST}}$ was converted into the corresponding attitude matrix D_{AST}. Then E_{AST}, the error in the AST-determined attitude, was computed as

$$E_{\text{AST}} = D_{\text{AST}} - D_0 \tag{12a}$$

Finally, the figure of merit was computed as the norm of E_{AST} as follows:

$$J_{\text{AST}} = \sqrt{\text{tr}\left(E_{\text{AST}} \cdot E_{\text{AST}}^T\right)} \tag{12b}$$

Note that the reason for this choice of figure of merit stemmed from

$$J_{\text{AST}} = \sqrt{\sum_{i=1}^{3} \sum_{j=1}^{3} e_{\text{AST}i,j}^2} \tag{12c}$$

where $e_{\text{AST}i,j}$ is the i, j element of the error matrix E_{AST}. Also note that the pairs $\boldsymbol{v}_{1,\text{meas}}$ and $\boldsymbol{u}_1$ and $\boldsymbol{v}_{2,\text{meas}}$ and $\boldsymbol{u}_2$ were computed just for the sake of simulating the attitude generated by the AST, namely, $\boldsymbol{q}_{\text{AST}}$. In other words, they were not a part of the algorithm.

Next we randomly chose $\boldsymbol{r}_3$, the unit vector to the sun in inertial (reference) coordinates, to be

$$\boldsymbol{r}_3 = \begin{bmatrix} -0.708 \\ 0.404 \\ 0.579 \end{bmatrix} \tag{13a}$$

and computed its corresponding vector in body coordinates,

$$\boldsymbol{b}_3 = D_0 \boldsymbol{r}_3 = \begin{bmatrix} 0.363 \\ 0.727 \\ 0.582 \end{bmatrix} \tag{13b}$$

We simulated the sun sensor measurement by adding errors to $\boldsymbol{b}_3$. Assuming that the sun sensor errors were 10 times greater than those of the star trackers, we computed $\boldsymbol{b}_{3,\text{meas}}$ by, first, adding an appropriate noise to $\boldsymbol{b}_3$ to obtain $\boldsymbol{b}_3'$:

$$\boldsymbol{b}_3' = \boldsymbol{b}_3 + \begin{bmatrix} \pm 10\varepsilon \\ \pm 10\varepsilon \\ \pm 10\varepsilon \end{bmatrix} \tag{13c}$$

and, then, normalizing $\boldsymbol{b}_3'$ to yield $\boldsymbol{b}_{3,\text{meas}}$. This ended the simulation of the AST-determined quaternion and of the sun sensor measurement.

Next we applied the new algorithm. First we computed

$$|\boldsymbol{e}_1 \times \boldsymbol{r}_3| = 0.706 \tag{14a}$$

$$|\boldsymbol{e}_2 \times \boldsymbol{r}_3| = 0.915 \tag{14b}$$

$$|\boldsymbol{e}_3 \times \boldsymbol{r}_3| = 0.815 \tag{14c}$$

Because the norm of the cross product $\boldsymbol{e}_2 \times \boldsymbol{r}_3$ yielded the largest value of the three, we chose $\boldsymbol{e}_2$ to be used in the computation of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ as follows:

$$\boldsymbol{r}_1 = \frac{\boldsymbol{e}_2 \times \boldsymbol{r}_3}{|\boldsymbol{e}_2 \times \boldsymbol{r}_3|} = \begin{bmatrix} 0.633 \\ 0 \\ 0.774 \end{bmatrix} \tag{15a}$$

$$\boldsymbol{r}_2 = \boldsymbol{r}_3 \times \boldsymbol{r}_1 = \begin{bmatrix} 0.313 \\ 0.915 \\ -0.225 \end{bmatrix} \tag{15b}$$

Next we computed the two vectors in the reference coordinates. This was done as follows:

$$\boldsymbol{b}_1 = D_b^r \boldsymbol{r}_1 = \begin{bmatrix} 0.904 \\ -0.426 \\ -0.033 \end{bmatrix} \tag{16a}$$

$$\boldsymbol{b}_2 = D_b^r \boldsymbol{r}_2 = \begin{bmatrix} 0.224 \\ 0.538 \\ -0.812 \end{bmatrix} \tag{16b}$$

Now we were ready to compute the attitude using the pseudovector measurements, that is, vectors 1 and 2, and the sun sensor measurement, vector 3. Because in this simulation we assumed that the accuracy of the sun sensor was 10 times worse than that of the star trackers in the AST, we computed the weights as follows. As indicated in Eqs. (6), we set $\alpha_1 = \alpha_2 = 1/\sigma_{\mathrm{ST}}^2$ and $\alpha_3 = 1/(10\sigma_{\mathrm{ST}})^2$ where σ_{ST} was the standard deviation of the star tracker measurement errors. The condition that the sum of the weights be 1 required the normalization presented in Eq. (7). This operation gave the results $\alpha_1 = \alpha_2 = \frac{100}{201}$ and $\alpha_3 = \frac{1}{201}$, which were, of course, independent of σ_{ST}. When these weights and the corresponding pairs $\boldsymbol{r}_1$ and $\boldsymbol{b}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{b}_2$, and $\boldsymbol{r}_3$ and $\boldsymbol{b}_3$ were used, the q method of Eqs. (8) and (9) was applied, which yielded the overall quaternion $\boldsymbol{q}_{\mathrm{ov}}$. The latter quaternion was then converted into the corresponding attitude matrix D_{ov}. Then E_{ov}, the error attitude matrix that corresponded to D_{ov}, was computed as

$$E_{\mathrm{ov}} = D_{\mathrm{ov}} - D_0 \tag{17a}$$

and the figure of merit that corresponded to E_{ov} was computed as follows:

$$J_{\mathrm{ov}} = \sqrt{\mathrm{tr}\left(E_{\mathrm{ov}} \cdot E_{\mathrm{ov}}^T\right)} \tag{17b}$$

Finally, to check the influence of the sun sensor measurement on the accuracy of the attitude determination, a ratio μ was computed as a ratio between the figure of merit after and before the addition of the sun sensor measurement, that is,

$$\mu = J_{\mathrm{ov}}/J_{\mathrm{AST}} \tag{18}$$

Numerous runs were made with these data. The difference in the input data between the various runs was the combinations of pluses and minuses in the errors [see Eqs. (11e), (11f), and (13c)]. It was found that, other than the rare cases where the attitude determination merit value J_{AST} was around 25×10^{-6}, its value was almost always between 200×10^{-6} and 400×10^{-6}. After the sun sensor update, the attitude determination merit value J_{ov} was almost always between 5×10^{-6} and 30×10^{-6}. As a result, almost always $\mu < 1$, that is, almost always adding the sun sensor measurement added to the accuracy. There were, however, cases where μ was slightly larger than 1. These were the mentioned cases where J_{AST} was around 25×10^{-6}, that is, the AST-generated attitude was so accurate that, to a small degree, the added sun sensor measurement spoiled that accuracy. The conclusion of this experiment was that even though the sun sensor accuracy was 10 times worse than that of the AST, it was worth using the sun sensor measurement because it almost always improved the accuracy considerably, and in the cases where it slightly worsened the accuracy, the latter was still so good that it did not matter. (Note that one cannot select to use only the favorable cases because in reality they are unknown to the user.)

Conclusions

In this Note, we presented an algorithm for optimal mixing of an AST-generated quaternion with a vector measured by another sensor. In the first step of the algorithm, the quaternion is replaced by two pseudovector measurements. These vectors are constructed perpendicularly to the vector measured by the other sensor and to one another. Finally, when the q-method is used and the appropriate weights are chosen, all three vectors are used to generate the overall attitude quaternion. Simulation runs were made that indicated that even though the sun sensor measurements were less accurate, it was desirable to use this measurement, too.

The algorithm presented here can be easily extended to the case of multiple ASTs and multiple vector-measuring devices.

References

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